Generalized Lorentzian integrals for small widths

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A typical textbook expression is that

$$\lim_{\eta\to 0}\frac{1}{x+i\eta} = \mathcal{P}\frac{1}{x} - i\pi\delta(x)$$

or more precisely under an integral with smooth and analytic $f(x)$

$$\lim_{\eta\to 0}\int dx\frac{f(x)}{x+i\eta} = \mathcal{P}\int dx\frac{f(x)}{x} - i\pi f(0)$$

Here we want to both justify this type of thing and work out the more general case of the real integral

$$I(n,p;A,B) = \lim_{\eta\to 0}\int_{-B}^{A} dx\frac{x^n f(x)}{(x^2+\eta^2)^p} \tag{1}$$

where n is a non-negative integer and $p \geq 0$. Here $A, B \gg \eta > 0$ are some "large" numbers that we may (or may not) send to infinity. Notice that by taking real and imaginary parts, $I(n,p)$ is a superset of what we have written above. This integral is a main focus, but a related integral that is half of it over positive x is also of interest

$$H(n,p;A) = \lim_{\eta\to 0}\int_{0}^{A} dx\frac{x^n f(x)}{(x^2+\eta^2)^p} \tag{2}$$

H is discussed after final forms for I are given.

An elementary way to proceed is to first split the integral into a region around the origin and outside of that. Specifically, let $\epsilon > 0$ and then split the integral

$$\begin{aligned} I(n,p;A,B) &= \lim_{\eta\to 0}\left(\int_{-B}^{-\epsilon} + \int_{-\epsilon}^{\epsilon} + \int_{\epsilon}^{A}\right) dx\frac{x^n f(x)}{(x^2+\eta^2)^p} \\ &= I(n,p;-\epsilon,B) + I(n,p;\epsilon,\epsilon) + I(n,p;A,-\epsilon)\,. \end{aligned}$$

We will be sending $\epsilon \to 0$ but in such a way so that $A, B \gg \epsilon \gg \eta$. The easiest thing is to set ϵ to a very small fixed number, which we will later (after sending $\eta \to 0$) consider sending to zero.

Let's first focus on the regions excluding the origin such as $I(n,p;A,-\epsilon)$

$$I(n,p;A,-\epsilon) = \lim_{\eta\to 0}\int_\epsilon^A dx \frac{x^n f(x)}{(x^2+\eta^2)^p}$$

We can rewrite the term in the integrand multiplying $f(x)$ as

$$\frac{x^n}{(x^2+\eta^2)^p} = \frac{x^{n-2p}}{(1+(\eta/x)^2)^p}$$

and since for this integral $0 < \epsilon \le x \le A$, This factor will approach x^{n-2p} with no problems as $\eta \to 0$. Then we can conclude that

$$I(n,p;A,-\epsilon) = \int_\epsilon^A dx f(x) x^{n-2p} \,. \tag{3}$$

The logic for the other integral $I(n,p;-\epsilon,B)$ is identical

$$I(n,p;-\epsilon,B) = \int_{-B}^{-\epsilon} dx f(x) x^{n-2p} \,. \tag{4}$$

Both are well behaved if $n-2p+1>0$ so the integral in the small x region is convergent. Let's be more explicit by series expanding $f(x)$:

$$\begin{aligned} I(n,p;A,-\epsilon)+I(n,p;-\epsilon,B) &= \left(\int_\epsilon^A + \int_{-B}^{-\epsilon}\right) dx f(x) x^{n-2p} \\ &= \left(\int_\epsilon^A + \int_{-B}^{-\epsilon}\right) dx \left(f(0)x^{n-2p} + f'(0)x^{n-2p+1}\right. \\ &\quad \left. + f''(0)x^{n-2p+2}/2 + \cdots\right) \end{aligned}$$

The main focus is on the possibly divergent parts coming from x close to ϵ, so we do the integrals from $\pm\epsilon$ to some places and only keep track of the ϵ dependence:

$$\begin{aligned} I(n,p;A,-\epsilon)+I(n,p;-\epsilon,B) &= f(0)\left[-\epsilon^{n-2p+1} + (-\epsilon)^{n-2p+1}\right]/(n-2p+1) \\ &\quad + f'(0)\left[-\epsilon^{n-2p+2} + (-\epsilon)^{n-2p+2}\right]/(n-2p+2) \\ &\quad + f''(0)\left[-\epsilon^{n-2p+3} + (-\epsilon)^{n-2p+3}\right]/(2(n-2p+3)) \\ &\quad + \text{constants} + O(\epsilon^{n-2p+4}) \end{aligned}$$

Now $(-1)^{2p}=1$ so we can simplify to

$$\begin{aligned} I(n,p;A,-\epsilon)+I(n,p;-\epsilon,B) &= -f(0)\epsilon^{n-2p+1}\left[1+(-1)^n\right]/(n-2p+1) \\ &\quad - f'(0)\epsilon^{n-2p+2}\left[1-(-1)^n\right]/(n-2p+2) \\ &\quad - f''(0)\epsilon^{n-2p+3}\left[1-(-1)^n\right]/(2(n-2p+3)) \\ &\quad + \text{constants} + O(\epsilon^{n-2p+4}) \end{aligned}$$

If an exponent or denominator $n-2p+q=0$ in any term , that term should be replaced by zero: this is because $x^{n-2p+q-1} = x^{-1}$ was the integrand which gives a logarithm $\ln x$

when integrated, and the contributions from $+\epsilon$ and $-\epsilon$ cancel because we get $-\ln\epsilon$ from the positive x integral and $\ln\epsilon$ from the negative x integral.

We are now ready to write a final form for the sum of these two integrals over the region $|x| \geq \epsilon$ in terms of their leading contribution as a function of ϵ. As long as $n-2p+1>0$, all of ϵ dependent terms will go to zero and there are no worries: the region about the origin is well behaved. So

$$I(n,p;A,-\epsilon) + I(n,p;-\epsilon,B) = \mathcal{P}\int_{\epsilon}^{A} dx f(x) x^{n-2p} \quad \text{if } n \geq 2p-1$$

If $n-2p+1$=0, as explained above there is no divergent contributions from the logarithms as they cancel. So we included that case in this formula as well.

Now for the more involved $n-2p+1<0$ cases. The leading contribution we care about has the lowest power of ϵ: when n is even, this is just $n-2p+1$, but if n is odd we have to move to the next term of power $n-2p+2$ and this power could be negative, zero, or positive, so we have to worry about all these possibilities. Here is the master table where ϵ is sent to zero if possible:

$$I(n,p;A,-\epsilon) + I(n,p;-\epsilon,B) = \begin{cases} \mathcal{P}\int_{-B}^{A} dx f(x) x^{n-2p} & \text{if } n \geq 2p-1 \\ -2f(0)\epsilon^{n-2p+1}/(n-2p+1) & \text{if } n < 2p-1 \;\&\; n \text{ even} \\ -2f'(0)\epsilon^{n-2p+2}/(n-2p+2) & \text{if } n < 2p-2 \;\&\; n \text{ odd} \\ \mathcal{P}\int_{-B}^{A} dx f(x) x^{n-2p} & \text{if } n \geq 2p-2 \;\&\; n \text{ odd} \end{cases}$$

Now we turn to the integral around the origin $I(n,p;\epsilon,\epsilon)$

$$I(n,p;\epsilon,\epsilon) = \lim_{\eta\to 0}\int_{-\epsilon}^{\epsilon} dx \frac{x^n f(x)}{(x^2+\eta^2)^p}$$

Here the integrand at $x=0$ becomes potentially divergent as the denominator denominator is η^{2p} which goes to zero. The first thing we can do is to note that since $\epsilon \to 0$ we can safely series expand $f(x)$ about the origin since we assume it is analytic:

$$I(n,p;\epsilon,\epsilon) = \lim_{\eta\to 0}\int_{-\epsilon}^{\epsilon} dx \; \frac{x^n f(0) + x^{n+1} f'(0) + x^{n+2} f''(0)/2 + \cdots}{(x^2+\eta^2)^p}$$

and the higher order terms are truly subdominant in a controlled manner. We can now rescale the integration coordinate to $y = x/\eta$ to get

$$I(n,p;\epsilon,\epsilon) = \lim_{\eta\to 0}\left\{\eta^{n-2p+1}\int_{-\epsilon/\eta}^{\epsilon/\eta} dy \; \frac{y^n f(0) + \eta y^{n+1} f'(0) + \eta^2 y^{n+2} f''(0)/2 + \cdots}{(y^2+1)^p}\right\}$$

For the moment, let's focus on a single term in the Taylor series and so we let $m \geq n$ be some integer and let's focus on

$$J_m = \eta^{n-2p+1} \int_{-\epsilon/\eta}^{\epsilon/\eta} dy \; \frac{\eta^{m-n} y^m}{(y^2+1)^p} = \eta^{m-2p+1} \int_{-\epsilon/\eta}^{\epsilon/\eta} dy \; \frac{y^m}{(y^2+1)^p}$$

which means that

$$I(n,p;\epsilon,\epsilon) = \lim_{\eta\to 0} \left\{ \sum_{m=n}^{\infty} \frac{J_m f^{(m)}(0)}{m!} \right\} = \lim_{\eta \to 0} \left\{ J_n f(0) + J_{n+1} f'(0) + J_{n+2} f''(0)/2 + \cdots \right\}$$

Note that m must be even or otherwise the integral is zero, so we assume m is even below, whence

$$J_m = 2\eta^{m-2p+1} \int_0^{\epsilon/\eta} dy \; \frac{y^m}{(y^2+1)^p}$$

The integrant is finite for $y \to 0$, so any problems will come from the large y region. For large y, the integrand is approximately y^{m-2p} so doing the integral for large y, we can write that for large ϵ/η and $m \neq 2p-1$

$$J_m = 2\eta^{m-2p+1} \cdot \left(\text{constant} + \frac{(\epsilon/\eta)^{m-2p+1}}{m-2p+1} + O((\epsilon/\eta)^{m-2p-1}) \right)$$

while for $m = 2p-1$ it is logarithmic

$$J_m = 2 \cdot \left(\text{constant} + \ln(\epsilon/\eta) + O((\eta/\epsilon)^2) \right)$$

This means that for some bounded constants C, D, the leading contributions are

$$J_m = \begin{cases} C\eta^{m-2p+1} + D\epsilon^{m-2p+1} & \text{if } m-2p+1 \neq 0 \\ 2\ln(\epsilon/\eta) & \text{if } m-2p+1 = 0 \end{cases}$$

This goes to zero for $m > 2p-1$ and is divergent for $m \leq 2p-1$ as $\eta \to 0$. In either case, however, we have a clear subdominant behavior for larger m: J_{m+2}/J_m scales either as ϵ^2 or η^2 for small η and ϵ and goes to zero.

We have learned that when $m \leq 2p-1$ the answer is formally infinite in the limit $\eta \to 0$ so if we want something to work with we had best keep η finite but small. But then the higher order terms are truly negligible in comparison.

To make progress, notice that the above inequalities show that the integral

$$\frac{J_m}{\eta^{m-2p+1}} = \int_{-\epsilon/\eta}^{\epsilon/\eta} dy \frac{y^m}{(y^2+1)^p}$$

is finite for $m < 2p-1$ as $\epsilon/\eta \to \infty$. The bad case is when $m = 2p-1$ exactly so that the integral is divergent like $\ln(\epsilon/\eta)$ due to the contributions at large y. Since $m \geq 0$ is even, this can only happen for $(m = 0, p = 1/2)$, $(m = 2, p = 3/2)$ and so forth.

So for η tending to zero and ϵ very small but fixed,

$$J_m = \begin{cases} D\epsilon^{m-2p+1} & \text{if } m > 2p-1 \\ \eta^{m-2p+1} \int_{-\infty}^{\infty} dy \, \frac{y^m}{(y^2+1)^p} & \text{if } m < 2p-1 \\ 2\ln(\epsilon/\eta) & \text{if } m = 2p-1 \end{cases}$$

Before writing down our final conclusions, we note that since $m \geq n$, J_n will be the dominant term anyways in the series without any trouble assuming n is even. If n is odd, then $J_n = 0$ and J_{n+1} is dominant. So we have that for small ϵ and η going to zero,

$$\int_{-\epsilon}^{\epsilon} dx \frac{x^n f(x)}{(x^2+\eta^2)^p} = \begin{cases} O(\epsilon^{n-2p+1}) & \text{if } n > 2p-1 \\ 2f(0)\ln(\epsilon/\eta) & \text{if } n = 2p-1 \;\&\; n \text{ even} \\ O(\epsilon) & \text{if } n = 2p-1 \;\&\; n \text{ odd} \\ f(0)\eta^{n-2p+1} Q(n,p) & \text{if } n < 2p-1 \;\&\; n \text{ even} \\ f'(0)\eta^{n-2p+2} Q(n+1,p) & \text{if } n < 2p-2 \;\&\; n \text{ odd} \\ 2f'(0)\ln(\epsilon/\eta) & \text{if } n = 2p-2 \;\&\; n \text{ odd} \\ O(\epsilon^{n-2p+2}) & \text{if } n > 2p-2 \;\&\; n \text{ odd} \end{cases}$$

Here

$$Q(n,p) \equiv 2\int_0^{\infty} dy \, \frac{y^n}{(y^2+1)^p} \, .$$

Here is the final collected result. The table lists the leading term only for small η and small ϵ. To get these results, we assume that η is much smaller than ϵ since we will be sending η to zero first and then ϵ.

$$\boxed{\int_{-B}^{A} dx \frac{x^n f(x)}{(x^2+\eta^2)^p} = \begin{cases} \mathcal{P}\int_{-B}^{A} dx f(x) x^{n-2p} & \text{if } n > 2p-1 \\ 2f(0)\ln(\epsilon/\eta) & \text{if } n = 2p-1 \;\&\; n \text{ even} \\ \mathcal{P}\int_{-B}^{A} dx f(x) x^{n-2p} & \text{if } n = 2p-1 \;\&\; n \text{ odd} \\ f(0)\eta^{n-2p+1} Q(n,p) & \text{if } n < 2p-1 \;\&\; n \text{ even} \\ f'(0)\eta^{n-2p+2} Q(n+1,p) & \text{if } n < 2p-2 \;\&\; n \text{ odd} \\ 2f'(0)\ln(\epsilon/\eta) & \text{if } n = 2p-2 \;\&\; n \text{ odd} \\ \mathcal{P}\int_{-B}^{A} dx f(x) x^{n-2p} & \text{if } n > 2p-2 \;\&\; n \text{ odd} \end{cases}}$$

We now turn to $H(n,p,A)$ which is

$$H(n,p,A) = \lim_{\eta\to 0}\int_0^A dx \frac{x^n f(x)}{(x^2+\eta^2)^p}$$

As before, we split it into two parts:

$$H(n,p,A) = I(n,p;A,-\epsilon) + I(n,p;\epsilon,0) = \int_\epsilon^A dx \frac{x^n f(x)}{(x^2+\eta^2)^p} + \int_0^\epsilon dx \frac{x^n f(x)}{(x^2+\eta^2)^p}$$

We have already derived the various expansions for these separate integrals: for the $x \geq \epsilon$ part we can send η to zero with no troubles

$$\int_\epsilon^A dx \frac{x^n f(x)}{(x^2+\eta^2)^p} = \int_\epsilon^A dx f(x) x^{n-2p}$$

We can then do the various cases where the main difference is we must keep the logarithmic case on hand for $n = 2p-1$:

$$I(n,p;A,-\epsilon) = \begin{cases} \mathcal{P}\int_0^A dx f(x) x^{n-2p} & \text{if } n > 2p-1 \\ -f(0)\ln(\epsilon) - f'(0)\epsilon + O(\epsilon^2) & \text{if } n = 2p-1 \\ -f(0)\epsilon^{n-2p+1}/(n-2p+1) - f'(0)\epsilon^{n-2p+2}/(n-2p+2) + \cdots & \text{if } n < 2p-1 \end{cases}$$

Similarly, we have for the $0 \leq x \leq \epsilon$ integral

$$I(n,p;\epsilon,0) = \lim_{\eta\to 0}\left\{\eta^{n-2p+1}\int_0^{\epsilon/\eta} dy\ \frac{y^n f(0) + \eta y^{n+1} f'(0) + \eta^2 y^{n+2} f''(0)/2 + \cdots}{(y^2+1)^p}\right\}$$

which gives

$$I(n,p;\epsilon,0) = \begin{cases} 0 & \text{if } n > 2p-1 \\ f(0)[\ln(\epsilon/\eta) + O((\eta/\epsilon)^2)] + f'(0)\epsilon + \cdots & \text{if } n = 2p-1 \\ f(0)\eta^{n-2p+1} Q(n,p)/2 + \cdots & \text{if } n < 2p-1 \end{cases}$$

Combining and only keeping leading terms

$$\boxed{\lim_{\eta\to 0}\int_0^A dx \frac{x^n f(x)}{(x^2+\eta^2)^p} = \begin{cases} \mathcal{P}\int_0^A dx f(x) x^{n-2p} & \text{if } n > 2p-1 \\ -f(0)\ln(\eta) & \text{if } n = 2p-1 \\ f(0)\eta^{n-2p+1} Q(n,p)/2 & \text{if } n < 2p-1 \end{cases}}$$

Examples:

1. The standard case of

$$\lim_{\eta\to 0}\int_{-B}^A dx \frac{f(x)}{x+i\eta} = \lim_{\eta\to 0}\int_{-B}^A dx \frac{(x-i\eta)f(x)}{x^2+\eta^2} = \lim_{\eta\to 0}\int_{-B}^A dx \frac{x f(x)}{x^2+\eta^2} - i\lim_{\eta\to 0}\eta\int_{-B}^A dx \frac{f(x)}{x^2+\eta^2}$$

whose real part is $(n=1,p=1)$ with $n=2p-1=1$ and n even, and imaginary part is η times the integral for $(n=0,p=1)$ with $n<2p-1$ and n even:

$$\lim_{\eta\to 0}\int_{-B}^{A}dx\frac{f(x)}{x+i\eta}=\mathcal{P}\int_{-B}^{A}dx\ \frac{f(x)}{x}-i\lim_{\eta\to 0}\eta\cdot f(0)\eta^{-1}Q(0,1)$$

Since $Q(0,1)=\pi$ we have just derived the usual result

$$\lim_{\eta\to 0}\int_{-B}^{A}dx\frac{f(x)}{x+i\eta}=\mathcal{P}\int_{-B}^{A}dx\ \frac{f(x)}{x}-i\pi f(0)$$

2. A real Lorentzian integral

$$\int_{-B}^{A}dx\frac{f(x)}{x^2+\eta^2}$$

Here $(n=0,p=1)$ with n even. The leading divergence is from η but we can also keep the ϵ divergence to be complete:

$$\int_{-B}^{A}dx\frac{f(x)}{x^2+\eta^2}=f(0)\pi/\eta-2f(0)/\epsilon$$

However, if $f(0)=0$ then we are in good shape! Let $g(x)=f(x)/x$ and we are assuming $g(x)$ is smooth around the origin so that $g(0)$ is finite. Then $f(x)=xg(x)$ and we are in the $(n=1,p=1)$, $n=2p-1$ and with n odd case so the integral is fine and well-behaved so

$$\int_{-B}^{A}dx\frac{xg(x)}{x^2+\eta^2}=\mathcal{P}\int_{-B}^{A}dx\ \frac{g(x)}{x}=\mathcal{P}\int_{-B}^{A}dx\ \frac{f(x)}{x^2}$$

3. The half integral

$$\int_{0}^{A}dx\frac{f(x)}{x^2+\eta^2}$$

where $(n=0,p=1)$ so $n<2p-1=1$. Looking up the H table, we get the divergent form

$$\int_{0}^{A}dx\frac{f(x)}{x^2+\eta^2}=\frac{f(0)\pi}{2\eta}$$

However, if $f(0)=0$, the divergence is greatly reduced. Working with $g(x)=f(x)/x$ which is assume smooth and non-zero around $x=0$, we now have an integral with $(n=1,p=1)$ involving g where $n=2p-1=1$

$$\int_{0}^{A}dx\frac{f(x)}{x^2+\eta^2}=-g(0)\ln(\eta)=-f'(0)\ln(\eta)\quad\text{when}\quad f(0)=0$$

4. The half integral (square of above integrand)

$$\int_{0}^{A}dx\frac{f(x)^2}{(x^2+\eta^2)^2}$$

where $(n = 0, p = 2)$ so $n < 2p - 1 = 3$. Looking up the H table, we get the divergent form

$$\int_0^A dx \frac{f(x)^2}{(x^2 + \eta^2)^2} = \frac{f(0)^2 Q(0,2)}{2\eta^3}$$

However, if $f(0) = 0$, the divergence is greatly reduced. Working with $g(x) = f(x)/x$ which is assume smooth and non-zero around $x = 0$, we now have an integral with $g(x)$ now with $(n = 2, p = 2)$ involving g where $n = 2 < 2p - 1 = 3$

$$\int_0^A dx \frac{f(x)^2}{(x^2 + \eta^2)^2} = -\frac{g(0)^2 Q(2,2)}{2\eta} = \frac{f'(0)^2 Q(2,2)}{2\eta} \quad \text{when} \quad f(0) = 0$$

5. An related integral of the form

$$\int_0^A dx \frac{f(x)^2}{[(x-a)^2 + \eta^2][(x-b)^2 + \eta^2]}$$

where $0 < a < b < A$. This is not of the form that we have described above, but it is the regions around $x = a$ and $x = b$ that generically give the really large contributions for $\eta \to 0$. Around $x = a$, the Lorentzian centered around $x = b$ is smooth and has a nice Taylor series, and vice versa. So at the most divergent order, we have

$$\int_0^A dx \frac{f(x)^2}{[(x-a)^2 + \eta^2][(x-b)^2 + \eta^2]} = \frac{f(a)^2 \pi}{\eta} + \frac{f(b)^2 \pi}{\eta}$$

If we have a zero at a or b by luck, e.g. if $f(a) = 0$, then

$$\int_0^A dx \frac{f(x)^2}{[(x-a)^2 + \eta^2][(x-b)^2 + \eta^2]} = -f'(a)^2 \ln \eta + \frac{f(b)^2 \pi}{\eta}$$

If both are zeros by extreme luck

$$\int_0^A dx \frac{f(x)^2}{[(x-a)^2 + \eta^2][(x-b)^2 + \eta^2]} = -[f'(a)^2 + f'(b)^2] \ln \eta$$