Decay of a discrete state into a continuum (tunneling)

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Contents

These notes are based on my own notes as well as the nice work of E. Kogan.[1]

The basic question is the following: we have a discrete state with energy ϵ. It is coupled to a continuum of states (continuous energy band). We start the system in the discrete state at

[1] E. Kogan, "Nonexponential decay via tunneling to a continuum of finite width", arxiv http://arxiv.org/abs/quant-ph/0609011 , Sept 2006.

time $t = 0$. How does the system evolve, and how does the system decay out of the discrete state into the continuum (or tunnel out)?

There are many physical realizations of this type of situation. Perhaps we have an atomic or molecular excited state that decays via photon emission or via emission of an electron (if the system is an anion and energy is above the ionization limit). Perhaps we have an electron starting in an atomic or molecular state on a surface or in the bulk of a material that then couples to the continuum of band states. Perhaps we create the electron on a particular atomic state via transition from a core state by X-ray absorption.

1 The model

We begin with our model. We have our discrete state $|d\rangle$ with energy ϵ. We label the continuum states $|k\rangle$ with energies ω_k. The coupling between them is V_k. So the Hamiltonian is

$$H = \epsilon|d\rangle\langle d| + \sum_k \omega_k|k\rangle\langle k| + \sum_k V_k|d\rangle\langle k| + V_k^*|k\rangle\langle d| \tag{1}$$

and the wave function is

$$|\psi(t)\rangle = g(t)|d\rangle + \sum_k b(k,t)|k\rangle\,. \tag{2}$$

The basis states $\{|d\rangle, |k\rangle\}$ are assumed orthonormal. Our initial condition is that

$$g(0) = 1 \quad , \quad b(k,0) = 0\,. \tag{3}$$

For convenience, we will assume $g(t) = b(k,t) = 0$ for $t < 0$. Normalization means

$$|g(t)|^2 + \sum_k |b(k,t)|^2 = 1\,.$$

We would like to solve the time-dependent Schrödinger equation ($\hbar = 1$)

$$i\partial_t|\psi(t)\rangle = H|\psi(t)\rangle$$

which in terms of expansion coefficients is

$$i\dot{g}(t) = \epsilon g(t) + \sum_k V_k^* b(k,t) \quad , \quad i\dot{b}(k,t) = \omega_k b(k,t) + V_k g(t)\,.$$

An interesting thing to note is that this type of Hamiltonian is in fact quite generic. Given a general Hamiltonian matrix, we can select one state (row/column) to be the "discrete" state $|d\rangle$ with diagonal element ϵ. The remainder of the Hamiltonian minus the row and column corresponding to $|d\rangle$ can then be diagonalized to yield the energies ω_k. The Hamiltonian matrix elements between $|d\rangle$ and $|k\rangle$ are then the V_k.

2 Eigenstates

Although it is not convenient to solve the problem in the eigenbasis, we will study this anyways since it provides useful information for what follows. An eigenstate $|E_j\rangle$ (indexed by j) has expansion coefficients obeying

$$E_j a_j = \epsilon a_j + \sum_k V_k^* c_{kj} \quad , \quad E_j c_{kj} = V_k a_j + \omega_k c_{kj} \, .$$

For this simple model, we can solve for c_{kj} explicitly:

$$c_{kj} = \frac{V_k}{E_j - \omega_k} a_j \, .$$

Plugging this into the equation for a_j gives a relation only in terms of a_j. Either $a_j = 0$ and we have nothing (i.e., E_j is not an eigenvalue afterall) or $a_j \neq 0$ and we must have the condition

$$E_j = \epsilon_j + \sum_k \frac{|V_k|^2}{E_j - \omega_k} \, .$$

This is the condition for an eigenvalue, and it has a simple graphical interpretation. For example, we are guaranteed to have one eigenvalue between pairs of adjacent ω_k. Thus is we have N values of ω_k that are monotonically increasing, then $N-1$ of the eigenvalues must lie between these; since the ω_k form a dense band, so must the eigenvalues which will pretty much have the same layout and density as the ω_k. Outside the band of ω_k we may or may not have solutions that separates off from the band by a finite amount; this depends on the V_k and the density of states of the band.

Next, normalization of the eigenstate

$$1 = |a_j|^2 + \sum_k |c_{kj}|^2$$

turns into a simple formula for $|a_j|^2$ once we plug in the form for c_{kj} which is

$$|a_j|^2 = \frac{1}{1 + \sum_k |V_k|^2/(E_j - \omega_k)^2} \, .$$

Since $a_j = \langle d|E_j\rangle$, this is the probability $|\langle d|E_j\rangle|^2$ of being in the discrete state $|d\rangle$ for eigenstate $|E_j\rangle$.

A final thing to observe is how $|a_j|^2$ behaves for a discrete state that occurs outside the band when we go to weak coupling $V_k \to 0$. If the discrete state does exist, say above the band, then as $V_k \to 0$ its energy must smoothly go to the top energy of the band ω_T. Thus in the eigenvalue condition, it must be that $|V_k|^2/(E_j - \omega_T)$ remains finite as $V_k \to 0$ and $E_j \to \omega_T^+$. Then $|V_k|^2/(E_j - \omega_T)^2$ must go to infinity and $|a_j|^2 \to 0$: as we weaken the coupling, either the bound state disappears or its projection onto the discrete state must to go zero.

3 Green's function solution and self-energy

The differential equation for $g(t)$ and $b(k,t)$ is not easy to solve as written, so we do a Fourier transform to change to an algebraic equation. Our Fourier convention is

$$g(\omega) = \int_{-\infty}^{\infty} dt\, g(t) e^{i\omega t} \quad , \quad g(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, g(\omega) e^{-i\omega t} \,. \tag{4}$$

The choice of initial conditions and zero wave function for $t < 0$ mean the time integrals for the Fourier transform run only over $t \geq 0$. So in our specific case

$$g(\omega) = \int_{0}^{\infty} dt\, g(t) e^{i\omega t}$$

and we can get this to absolutely converge if $Im\ \omega > 0$ because $|g(t)| \leq 1$ by normalization. Doing the Fourier transform and using $g(0) = 1$ and $b(k,0) = 0$, we get

$$-i + \omega g(\omega) = \epsilon g(\omega) + \sum_k V_k^* b(k,\omega) \quad , \quad \omega b(k,\omega) = \omega_k b(k,\omega) + V_k g(\omega) \,.$$

We can easily eliminate $b(k,\omega)$ and solve for $g(\omega)$ alone:

$$g(\omega) = \frac{i}{\omega - \epsilon - \Sigma(\omega)} \tag{5}$$

where the self-energy or mass operator is

$$\Sigma(\omega) = \sum_k \frac{|V_k|^2}{\omega - \omega_k} \,.$$

We note that even though we didn't have any interactions to speak of in the usual sense of the word, we have gotten rid of degrees of freedom (all the $b(k,t)$) so the effective dynamics of $g(t)$ alone must feel a more complex and time-dependent "potential" which is the self-energy.

At this point it is important to note the connection between $g(\omega)$ as written above and the local density of states for $|d\rangle$. Since it is also true that

$$g(t) = \langle d| \exp(-i\hat{H}t) |d\rangle = \sum_j |\langle d|E_j\rangle|^2 \exp(-iE_j t) \,.$$

we can do the same type of Fourier transform to get

$$g(\omega) = \sum_j \frac{|\langle d|E_j\rangle|^2\, i}{\omega - E_j + i0^+}$$

from which we have the local density of states projected onto $|d\rangle$ or spectral function

$$A(\omega) = -\frac{1}{\pi} Im\, [g(\omega)/i] = \sum_j |\langle d|E_j\rangle|^2 \delta(\omega - E_j) = \frac{-Im\Sigma(\omega)}{(\omega - \epsilon - Re\Sigma(\omega))^2 + (Im\Sigma(\omega))^2} \,.$$

We would like to take the continuum limit as the sum over k becomes infinitely dense and the ω_k infinitesimally separated. But the present form is slightly inconvenient so we first rewrite $\Sigma(\omega)$ in terms of a hybridization function $\Delta(E)$ for continuous energy E

$$\Delta(E) = \sum_k |V_k|^2 \delta(E - \omega_k)\,. \tag{6}$$

Notice how this function contains all the information about the coupling to the band as well as the discreteness of the band itself. In addition, assuming V_k is "smooth" versus ω_k, the function $\Delta(E)$ converges to a continuous function of E in a straightforward manner. The self-energy now can be written as a continuous integral

$$\Sigma(\omega) = \int_{\omega_B}^{\omega_T} dE\, \frac{\Delta(E)}{\omega - E} \tag{7}$$

where $\omega_T = \max \omega_k$ is the top of the band and $\omega_B = \min \omega_k$ is the bottom of the band.

4 Analytical behavior of Σ and its continuation

We now need to analyze the analytical properties of $\Sigma(\omega)$ and set up the inverse Fourier integral to get $g(t)$, which is again

$$g(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \frac{ie^{-i\omega t}}{\omega - \epsilon - \Sigma(\omega)}\,. \tag{8}$$

We want to evaluate this integral for $t > 0$. To use contour integral methods and create a closed loop, $t > 0$ means that $Im\ \omega < 0$ is needed to get exponential convergence as $\omega \to \infty$ so we would close this with a loop over a half circle on the lower half plane. However, our derivation and definition of $\Sigma(\omega)$ so far has assumed $Im\ \omega > 0$: namely, we can only perform the above Fourier integral for the imaginary part of ω being slightly positive. So we need to first figure out how to define things in the lower complex ω half plane. Namely, we need to analytically continue $\Sigma(\omega)$ to the lower half plane.

First, let's see what is going on around the segment of continuous band energies $\omega_B < \omega < \omega_T$. Namely, let's approach an ω in that segment from above and below the real axis, so $\omega = \omega_1 + is$ where ω_1, s are real, $\omega_B < \omega_1 < \omega_T$ and s is going to zero either from positive or negative side. Writing this out

$$\Sigma(\omega_1 + is) = \int_{\omega_B}^{\omega_T} dE\, \frac{\Delta(E)}{\omega_1 - E + is}$$

Remembering that

$$\lim_{s\to 0} \frac{1}{x + is} = P\frac{1}{x} - i\pi\, \mathrm{sign}(s)\, \delta(x)$$

we find for $s \to 0$

$$\Sigma(\omega_1 + is) = P\int_{\omega_B}^{\omega_T} dE\, \frac{\Delta(E)}{\omega_1 - E} - i\pi\, \mathrm{sign}(s) \Delta(\omega_1)\,.$$

Therefore we have a branch cut for $\omega_B < \omega < \omega_T$ since the imaginary part is discontinuous. So we will have to integrate around this non-analyticity. For what follows, we separate out the real and imaginary part of $\Sigma(\omega)$ for ω close to the real axis:

$$\begin{aligned}\Sigma(\omega_1 \pm i0^+) &= \Sigma_1(\omega_1) \pm i\Sigma_2(\omega_1) \\ \Sigma_1(\omega_1) &= P\int_{\omega_B}^{\omega_T} dE\,\frac{\Delta(E)}{\omega_1 - E} \\ \Sigma_2(\omega_1) &= -\pi\Delta(\omega_1)\,.\end{aligned} \tag{9}$$

Second, when doing the Fourier integral to get $g(t)$, it could turn out that $\omega - \epsilon - \Sigma(\omega) = 0$ for some ω outside of the branch cut. Namely a bound (or anti-bound) state would appear below (or above) the continuum band energy range. This depends on the precise form of the $\Delta(E)$, but if they exist, they will be a countable set of solutions. The reason is based on our eigenstate analysis above and the graphical analysis: we have the eigenvalue condition

$$E_j = \epsilon + \sum_k \frac{|V_k|^2}{E_j - \omega_k}\,. \tag{10}$$

For a fixed number of $|k\rangle$ states N_k, this problem has $N_k + 1$ eigenvalues since that is the size of the Hamiltonian matrix. Doing the graphical analysis, we find there must be a solution between each consecutive pair of ω_k since the right hand side will go from $-\infty$ to $+\infty$ in that narrow region. So there can be at most two solutions that are separated from the band, one above and one below; whether they will be or will not be separated by a finite amount from the band as the continuum limit is approached depends on $\Delta(E)$.

So let us call these potential real discrete energy solutions E_j where $E_j = \omega - \epsilon - \Sigma(E_j)$ and the residue of the integrand for them is

$$Z_j = \lim_{\omega \to E_j} (\omega - E_j) \cdot \frac{1}{\omega - \epsilon - \Sigma(\omega)} = \frac{1}{1 - \Sigma'(E_j)}\,. \tag{11}$$

We can write this in more detail as

$$Z_j = \frac{1}{1 + \sum_k |V_k|^2/(E_j - \omega_k)^2}\,.$$

As noted in our eignstate analysis above, this just says that

$$|\langle d|E_j\rangle|^2 = Z_j \tag{12}$$

which means the residue is the probability/weight of the discrete state in the eigenstate $|E_j\rangle$ or it is the overlap of the initial state $|d\rangle$ with the discrete eigenstate $|E_j\rangle$.

Now we go back to our Fourier integral of Eq. (8). We have two types of non-analytic behavior in the integrand. There is a branch cut for $\omega_B < \omega < \omega_T$ and there may be some discrete and isolated poles at E_j to the left and/or right of the branch cut along the real ω axis. So when we close the contour integral over a infinite semicircle on the lower half plane, we find that our Fourier integral collapses to closed loops around the isolated poles at E_j and a closed integral around the branch cut going rightwards above it and leftwards below it. The isolated poles are simple to deal with since they just contribute the residues Z_j. The branch cut requires some more algebra. Looking back at the real and imaginary part of Σ in Eq. (9) close to the branch cut, we need to sum (above branch cut minus below it)

$$\int_{\omega_B}^{\omega_T} \frac{d\omega}{2\pi} \frac{ie^{-i\omega t}}{\omega - \epsilon - \Sigma(\omega + i0^+)} - \int_{\omega_B}^{\omega_T} \frac{d\omega}{2\pi} \frac{ie^{-i\omega t}}{\omega - \epsilon - \Sigma(\omega - i0^+)} .$$

Plugging in the explicit from for Σ and doing basic algebra we arrive at the result of the integral around the branch cut

$$\int_{\omega_B}^{\omega_T} \frac{d\omega}{2\pi} \frac{2\pi\Delta(\omega)e^{-i\omega t}}{(\omega - \epsilon - \Sigma_1(\omega))^2 + \pi^2\Delta(\omega)^2} .$$

Thus in final form we have for $g(t)$

$$g(t) = \sum_j Z_j e^{-iE_j t} + \int_{\omega_B}^{\omega_T} \frac{d\omega}{2\pi} \frac{2\pi\Delta(\omega)e^{-i\omega t}}{(\omega - \epsilon - \Sigma_1(\omega))^2 + \pi^2\Delta(\omega)^2} . \tag{13}$$

So we get the contribution from the discrete states which are just oscillatory and do not damp away, and we have a continuous integral that must go to zero as $t \to +\infty$ by the Riemann-Lebesgue lemma. Therefore, for long times, the discrete state decays into some component of the bound states but the rest of it "disappears" into the continuum.

For very large times $t \to \infty$, the integral will go to zero, but we can analyze the way it goes to zero. Throughout the range of integration (away from the band edges), we expect $\Delta(E)$ to be smooth and therefore $\Sigma_1(E)$ will be smooth as well. So the integrand is actually smooth in the interior of the band, so a Fourier transform of it will go to zero rapidly for large t. However, generally, there must be some type of discontinuity in $\Delta(E)$ across the band edge since it must be strictly zero outside the band. Let's focus for argument's sake on the lower band edge ω_B. If we assume $\Delta(E)$ behaves like $\sim (E - \omega_B)^\beta$ for some power $\beta \geq 0$ (so that $\Delta(E)$ is not divergent at the band edge), then simple scaling arguments (or analysis of the discontinuity in the β^{th} derivative of $\Delta(E)$) show that we get a contribution going to zero as $t^{-(\beta+1)}$. The upper band will also give power law decay. Thus the long time behavior of the contribution from the band goes to zero like a power law.

Eq. (13) is as far as we can go without known more about $\Delta(E)$. Below follow some examples.

5 Example: infinite band and constant coupling

The first example is the textbook example of a uniform continuum of infinite extent with uniform coupling to all continuum states. So $\omega_T = -\omega_B = \infty$ and $V_k = v$ so $\Delta(E) = \Delta_0$.

The real part of the self-energy in this case is actually zero ($z = \omega - E$ below)

$$\Sigma_1(\omega) = P\int_{-\infty}^{\infty} dE \frac{\Delta_0}{\omega - E} = \Delta_0 \lim_{u\to 0}\lim_{A\to\infty} \int_{-A}^{-u} \frac{dz}{z} + \int_u^A \frac{dz}{z} = 0\,.$$

So $g(t)$ in this case is given by a known Lorenzian integral, and there are no bound states since the continuum exists is for all energies, which means

$$g(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{2\pi\Delta_0 e^{-i\omega t}}{(\omega-\epsilon)^2 + \pi^2\Delta_0^2} = \exp(-\pi\Delta_0 t).$$

The probability is

$$p(t) = |g(t)|^2 = \exp(-2\pi\Delta_0 t)$$

so we have a lifetime $\tau = 1/(2\pi\Delta_0)$. This is classic exponential decay of the probability from the discrete state into the continuum. The density of states on the discrete state is essentially $g(\omega)$ which in this case is

$$g(\omega) = \frac{i}{(\omega-\epsilon)^2 + \pi^2\Delta_0^2}$$

so the discrete level has become broadened due to the coupling to the continuum and this is the same as saying it has acquired a finite lifetime.

6 Weak coupling and finite width band

In the perturbative limit where $|V_k|$ and thus $\Delta(E)$ is very small, the integral over the band is very sharply peaked around $\omega = \epsilon + \Sigma_1(\omega)$. Given that Σ_1 is small as well, this happens around $\omega = \epsilon$ but we can be more correct by doing a series expansion in ω around ϵ: the solution ω_s obeys

$$\omega_s \approx \epsilon + \Sigma_1(\epsilon) + (\omega_s - \epsilon)\Sigma_1'(\epsilon)$$

which means

$$\omega_s \approx \epsilon + \frac{\Sigma_1(\epsilon)}{1 - \Sigma_1'(\epsilon)} \approx \epsilon + \Sigma_1(\epsilon)$$

since we are only keeping lowest order terms in the small Σ quantity (which are proportional to the small V_k quantities). For $\Delta(\omega)$ we just evaluate it at ϵ since it already enters are squared which is second order. We also note that the function $\omega - \epsilon - \Sigma_1(\omega)$ around ω_s has the approximate linearized form $(\omega - \omega_s)(1 - \Sigma_1'(\epsilon)) = (\omega - \omega_s)/Z(\epsilon)$ where we have defined

$$Z(\epsilon) = \frac{1}{1 - \Sigma_1'(\epsilon)}\,.$$

Thus our integral is approximated as

$$g(t) \approx \sum_j Z_j e^{-iE_j t} + \int_{\omega_B}^{\omega_T} \frac{d\omega}{2\pi} \frac{2\pi\Delta(\omega)e^{-i\omega t}}{(\omega-\omega_s)^2/Z(\epsilon)^2 + \pi^2\Delta(\epsilon)^2}\,.$$

If $\Delta(\epsilon)$ is quite tiny, then the integrand is extremely peaked and gives a state with long lifetime; only with very long times does that contribution decay and we will see the effects of the band edges. So there are conditions when we will have something like

$$g(t) \approx \sum_j Z_j e^{-iE_j t} + Z(\epsilon) e^{-\pi\Delta(\epsilon)Z(\epsilon)t} e^{-i[\epsilon+\Sigma_1(\epsilon)]t} + At^{-(\beta_B+1)} + Bt^{-(\beta_T+1)} .$$

The first sum is from the discrete pole term: they scale as Δ for small Δ from the basic analysis above of the discrete poles for $\Delta \to 0$. The second term is from our sharp peak in the integral: it is the usual exponential decay from the discrete state into the band but renormalized by $Z(\epsilon)$. The last two terms coming from the possible discontinuities in Δ at the band edges also scale as Δ. So for weak Δ, for shorter times we see the exponential decay as in the infinite band width case but at longer times when this exponent becomes very small, we see algebraic decay from band edges and constant small amplitude oscillations from the potential bound states.

In this type of model $Z(\epsilon) > 1$ is often true so we can't interpret this as a usual renormalization that has $Z < 1$. Examination of the integral defining $\Sigma_1(\omega)$

$$\Sigma_1(\omega) = P\int_{\omega_B}^{\omega_T} dE \frac{\Delta(E)}{\omega - E}$$

shows the following facts: (i) since $\Delta \geq 0$ then Σ_1 must be negative for $\omega < \omega_B$ and positive for $\omega > \omega_T$; (ii) it drops down to zero as ω moves to larger values from ω_T; (iii) it rises to zero as ω moves to smaller values from ω_B; (iv) for very large $|\omega|$ outside the band $\Sigma_1 = C/\omega$ where $C > 0$. Therefore, we must have at last one zero of Σ_1 inside the band and we generically expect it to be increasing inside the band so $\Sigma_1' > 0$ is generically true inside the band and certainly must be true on overage inside the band. Conversely, $\Sigma_1' < 0$ outside the band for sure. Thus $Z > 1$ inside the band and $Z < 1$ outside. Of course, in some parts of the band, often near the band edges (if $\Delta \to 0$ near the edges), we may have $Z < 1$.

7 Finite band with constant coupling

A specific example is provided by a finite band of width $2W$ with uniform coupling:

$$\Delta(E) = \Delta_0 \qquad , \qquad \omega_T = -\omega_B = W .$$

This can be physically realized for a 2D system which has constant density of states (for parabolic energies of electrons) and a uniform coupling $|V_k|$ being constant.

Then we can perform the integral for $\Sigma_1(\omega)$ to get

$$\Sigma_1(\omega) = \Delta_0 \ln\left(\frac{|\omega + W|}{|\omega - W|}\right)$$

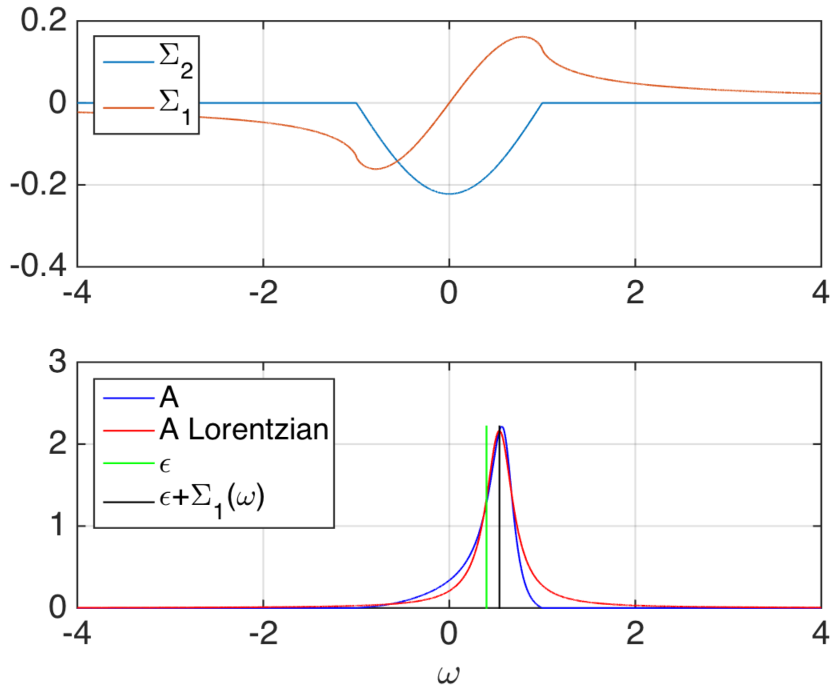

Figure 1: Finite width band model with $\Delta(\omega) = (\pi/4)v_0^2 \cos(\pi\omega/2)$. This specific case is for $v_0 = 0.3$ and $\epsilon = 0.4$. Top panel shows $\Sigma_1(\omega)$ and $\Sigma_2(\omega)$. Bottom panel shows in green vertical line ϵ and in black vertical line the position of $\hat{\omega}$ that solves $\epsilon + \Sigma_1(\hat{\omega}) = \hat{\omega}$; the difference is the energy shift due to the coupling. The blue curve A is the spectral function $\Delta/((\omega - \epsilon - \Sigma_1)^2 + \pi^2\Delta^2) = (-1/\pi)Im[g(\omega)/i]$ while the red curve is its approximation by the Lorentzian at $\hat{\omega}$.

and the bound state equation for E_j outside the band reads ($|E_j| > W$)

$$E_j = \epsilon + \Delta_0 \ln\left(\frac{|E_j + W|}{|E_j - W|}\right)$$

which must have two solutions outside the band because the logarithm diverges at the band edges and drops from infinity down to zero (very slowly). The transcendental equation is hard to solve, but here are some examples tabulated numerically:

ϵ/W	Δ_0/W	E_j/W	Z_j
0.1	1.0	1.59	0.43
0.5	1.0	1.78	0.52
0.9	1.0	2.00	0.60
0.1	0.5	1.23	0.34
0.5	0.5	1.40	0.49
0.9	0.5	1.62	0.62
0.1	0.2	1.02	0.093
0.5	0.2	1.10	0.35
0.9	0.2	1.30	0.64
0.1	0.1	1.0002	0.0003
0.5	0.1	1.01	0.11
0.9	0.1	1.16	0.63

In the perturbative limit of small Δ_0/W, we can also try to solve this. If we let $E_j = W(1+\delta)$ the equation now reads

$$1 + \delta = \epsilon/W + (\Delta_0/W) \ln\left(\frac{2}{\delta} + 1\right)$$

Assuming δ is small, to leading order the equation reads

$$\frac{W - \epsilon}{\Delta_0} \approx \ln(2/\delta)$$

which solves to

$$E_j \approx W + 2W \exp\left(-\frac{W - \epsilon}{\Delta_0}\right)$$

which is a very strong and non-perturbative dependence on Δ_0. Since

$$\Sigma_1'(\omega) = \frac{\Delta_0}{\omega + W} - \frac{\Delta_0}{\omega - W}$$

when evaluated at E_j we get

$$\Sigma_1'(E_j) \approx -\frac{\Delta_0}{2W} \exp\left(+\frac{W - \epsilon}{\Delta_0}\right)$$

and so Z_j to leading order is

$$Z_j \approx \frac{2W}{\Delta_0} \exp\left(-\frac{W - \epsilon}{\Delta_0}\right)$$

Thus the bound state weight Z_j becomes exponentially small for weak Δ_0. So in this limit, the integral over the band dominates the decay probability.

8 Other examples

Other examples and numerical examples can be found in the Kogan work cited at the start.