A dielectric slab: potential from a point charge in the middle

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We are interested in the dielectric screening properties of a dielectric slab. In the far field, it turns out that such a slab is rather ineffective at screening actually. We quantify this below.

We have a slab of infinite extent in the xy plane and finite thickness d in the z direction. The dielectric constant is ϵ in the slab and 1 outside (vacuum). We will have the slab be in the region $-d/2 < z < d/2$. So the dielectric function is

$$\epsilon(z) = \begin{cases} 1 & \text{for} \quad d/2 < z \\ \epsilon & \text{for} \quad -d/2 < z < d/2 \\ 1 & \text{for} \quad z < -d/2 \end{cases} .$$

We want to solve the classical electrostatic problem

$$\nabla \cdot (\epsilon(z)\nabla\phi) = -4\pi\rho_{free}$$

where the free charge can either be a unit point charge in the center of the slab at $z = 0$ or on the surface at $z = d/2$:

$$\rho_{free}(x, y, z) = \delta(x)\delta(y)\delta(z - z_0) .$$

The far fields are quite similar but having the two cases gives us some different insights. We obviously have cylindrical symmetry.

Aside from the boundaries at $z = \pm d/2$ and at $(x, y, z) = (0, 0, z_0)$, we have the usual Laplace equation. We will work in Fourier space in the xy plane but keep the real space z description. Thus the potential will be in mixed variables $\tilde{\phi}(k_x, k_y, z)$ and we will denote $k = (k_x, k_y)$. In these variables, $\tilde{\phi}$ is easy to solve for in regions where $\epsilon(z)$ is constant:

$$\tilde{\phi} = \alpha e^{kz} + \beta e^{-kz}$$

for some constants α and β. Thus we now have a boundary problem of matching the value and derivatives of $\tilde{\phi}$ at the three boundaries at hand. Also, we assume the potential goes to zero for $z \to \pm\infty$. Thus, we will therefore be seeking a solution of the form

$$\tilde{\phi}(k, z) = \begin{cases} D_2 e^{-kz} & \text{for} \quad d/2 < z \\ C_1 e^{kz} + C_2 e^{-kz} & \text{for} \quad z_0 < z < d/2 \\ B_1 e^{kz} + B_2 e^{-kz} & \text{for} \quad -d/2 < z < z_0 \\ A_1 e^{kz} & \text{for} \quad z < -d/2 \end{cases} .$$

The continuity conditions on $\tilde{\phi}$ are

$$C_1 + C_2 e^{-kd} = D_2 e^{-kd} \quad , \quad C_1 e^{kz_0} + C_2 e^{-kz_0} = B_1 e^{kz_0} + B_2 e^{-kz_0} \quad , \quad B_1 e^{-kd} + B_2 = A_1 e^{-kd}$$

while the derivative discontinuity conditions at some z come from the general relation

$$\epsilon(z^+) \frac{d\tilde{\phi}}{dz}\Big|_{z^+} - \epsilon(z^-) \frac{d\tilde{\phi}}{dz}\Big|_{z^-} = \begin{cases} -4\pi & \text{if } z = z_0 \\ 0 & \text{else} \end{cases}$$

so we have

$$\begin{aligned} -D_2 e^{-kd} - \epsilon(C_1 - C_2 e^{-kd}) &= 0 \\ C_1 - C_2 e^{-2kz_0} - B_1 + B_2 e^{-2kz_0} &= -\frac{4\pi e^{-kz_0}}{\epsilon k} \\ \epsilon(B_1 e^{-kd} - B_2) - A_1 e^{-kd} &= 0 \end{aligned}$$

Rather than solving this for the most general z_0, we will simply concentrate on the two cases of interest to us.

1. The case $z_0 = 0$ so the point charge in inside the center of the slab. Plugging in the above into Mathematica with $z_0 = 0$ gives

$$\begin{aligned} C_1 &= \frac{2\pi}{k} \cdot \frac{(\epsilon - 1)e^{-kd}}{\epsilon(\epsilon + 1 - (\epsilon - 1)e^{-kd})} \\ C_2 &= \frac{2\pi}{k} \cdot \frac{(\epsilon + 1)}{\epsilon(\epsilon + 1 - (\epsilon - 1)e^{-kd})} \\ D_2 &= \frac{2\pi}{k} \cdot \frac{2}{\epsilon + 1 - (\epsilon - 1)e^{-kd}} \end{aligned}$$

The shared denominator goes from 2 at $k = 0$ to $\epsilon + 1$ at $k \to \infty$.

The potential is given by the Fourier integral

$$\phi(x, y, z = 0) = \int \frac{dk_x dk_y}{(2\pi)^2} e^{ik_x x + ik_y y} \tilde{\phi}(k, z) = \int_0^\infty dk \, \frac{k}{2\pi} \, \tilde{\phi}(k, z) J_0(k\rho)$$

where $\rho = \sqrt{x^2 + y^2}$ and we have used cylindrical symmetry. Here is a very useful integral: and

$$\int_0^\infty dk \, e^{-k|z|} J_0(k\rho) \;\rightarrow\; \phi(\rho, z) = \frac{1}{\sqrt{\rho^2 + z^2}} \quad \text{(Bare Coulomb)}\,.$$

We are now ready to answer some questions. The first is the field inside the slab, $0 < z < d/2$, and close to the point charge. Namely, ρ is small and so is z (both compared to d of course). The integral to do is for the field at z given by $C_1 e^{kz} + C_2 e^{-kz}$ in k space:

$$\phi(\rho, 0 < z < d/2) = \int_0^\infty dk \, \frac{(\epsilon - 1)e^{-k(d-z)} + (\epsilon + 1)e^{-kz}}{\epsilon(\epsilon + 1 - (\epsilon - 1)e^{-kd})} J_0(k\rho)\,.$$

Since $z, \rho \ll d$, the denominator will have converged to $\epsilon(\epsilon + 1)$ before the numerator starts doing anything and before the J_0 starts oscillating. So the dominant contribution to the integral is from large k and we can just set the denominator to its value for $k \to \infty$. Also in the numerator, the first exponential will be quite small and can be neglected. So we end up with the integrand of $e^{-kz}J_0(k\rho)/\epsilon$ which then simply gives a screened Coulomb interaction:

$$\phi(\rho \ll d, z \ll d) = \frac{1}{\epsilon\sqrt{\rho^2 + z^2}}\,.$$

This is sensible: close to the point charge, it is as if we in an infinite 3D medium and the electric fields coming out of the point charge are screened isotropically in all directions as expected. The positive unit charge has been screened by negative polarization charges immediately around it, and the compensating positive polarization charges are on the surface of the slab and quite far away.

Second, we can ask for the far field. One possibility is to send $\rho \to \infty$ while keeping $0 < z < d/2$ in the slab; another is to send $z \to \infty$ out of the slab for fixed ρ; there are more but they all give the same answer actually. For fixed z but $\rho \gg d$, the region contributing to the integral is for $k \ll 1/\rho$ due to the oscillatory J_0. For small k, the numerator becomes 2ϵ and the denominator 2ϵ as well: we just get a bare Coulomb interaction $1/\rho$. For fixed ρ but large z, we need to do a different integral involving D_2:

$$\phi(\rho, z > d/2) = \int_0^\infty dk\, \frac{2e^{-kz}}{\epsilon(\epsilon + 1 - (\epsilon - 1)e^{-kd})} J_0(k\rho)\,.$$

Again only $k \ll 1/z$ gives significant contributions, we can set the J_0 to unity, the denominator to 2, and the integrand is really simple and again gives the Bare Coulomb interaction $1/\sqrt{\rho^2 + z^2}$. Therefore, the slab can not screen the Coulomb interaction in the far field. This is also straightforward: being a dielectric, only dipoles and no monopoles can be induced by the point charge. Assuming these charges are localized (which we will show) around the point charge, they can give only dipolar contributions in the far field which are subleading compared to the bare Coulomb contribution.

The near field analysis showed that a net negative polarization charge of $1/\epsilon - 1$ surrounds the point charge. So the positive polarization charge $1 - 1/\epsilon$ will be on the two surfaces of the slab (in the interior of the slab, $\nabla^2\phi = 0$ so there are no charges at all in the interior). How is this charge distributed? By symmetry we can just look at one surface. The surface charge density at $z = d/2$ is given by the discontinuity of $-d\tilde{\phi}/dz$ at $z = d/2$ divided by 4π. Some algebra shows that it is

$$\tilde{\Sigma}(k) = \frac{(\epsilon - 1)e^{-kd/2}}{\epsilon(\epsilon + 1 - (\epsilon - 1)e^{-kd})} = \frac{(1 - 1/\epsilon)}{2} \cdot \frac{e^{-kd/2}}{1 + (\epsilon - 1)(1 - e^{-kd})/2}\,.$$

We see that $\tilde{\Sigma}(k \to 0) = (1 - 1/\epsilon)/2$ which is the right amount of total polarization charge on the surface (half on top and half on bottom). The actual distribution is

$$\Sigma(\rho) = \int_0^\infty dk \frac{k}{2\pi} \tilde{\Sigma}(k) J_0(k\rho)\,.$$

An elementary integral for this case is

$$\tilde{\Sigma}(k) = e^{-kw} \quad \to \quad \Sigma(\rho) = \frac{w}{(w^2 + \rho^2)^{3/2}}$$

which is a unit of charge spread over a region of $\rho \sim w$. Naively, for our case, as long as $\rho \gg d$ such that the denominator is constant for $k \ll d$, we basically have the charge $(1 - 1/\epsilon)/2$ spread over a region of $\rho \sim d/2$ — the actual distribution is more complex for $\rho \sim d$ due to the more complex behavior of the integrand denominator.

We can do some more analysis for $\tilde{\Sigma}(k)$. Let's first series expand the k dependent part for weak screening $\epsilon \to 1$:

$$\tilde{\Sigma}(k) = \frac{(1-1/\epsilon)}{2} \cdot e^{-kd/2} \cdot \left[1 - \left(1 - e^{-kd}\right) \cdot \left(\frac{\epsilon - 1}{2}\right) + O\left(\frac{\epsilon-1}{2}\right)^2\right]$$

The leading term is just the charge spread over a region of size $d/2$, the next is the superposition of two opposite charges (a dipole) over regions of somewhat different size $\sim d$, etc. So basically the surface charge is spread out over a region of extent $\sim d$. The opposite limit of strong (metallic) screening $\epsilon \to \infty$ is more tricky as a direct expansion in $1/\epsilon$ gives divergent integrands. Instead, we focus on the rapidly varying denominator

$$1 + (\epsilon - 1)(1 - e^{-kd})/2 \,.$$

This function is 1 at $k = 0$, very large $\approx \epsilon/2$ for $k \gg d$, and by series expanding the exponential the transition happens over a region in k satisfying $kd\epsilon/2 \sim 1$ of $k \sim 2/(d\epsilon)$. For $\epsilon \gg 1$, this region in k is much smaller than $1/d$ so the numerator $e^{-kd/2}$ is basically constant over this region. So the integrand has the fastest variation over a narrow range of k of size $\sim 2/(d\epsilon)$ which corresponds in real space to $\rho \sim d\epsilon/2$. Thus for a almost metallic system, the charge is spread not over a region of size d on the surface but instead of much larger size ϵd — our native analysis above is actually only true for ρ larger than ϵd and not merely d. That the charge is more spread out makes sense: if we had a true metal, the positive surface charge would be completely spread out over the surface to give a constant potential everywhere (the point charge would be perfectly screened by negative charges in the interior).

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2. Now we put the point charge on the surface of the slab. Here it is more convenient to have $z = 0$ be the top surface of the slab where the point charge is located and have $z = -d$ be the bottom surface. Also, instead of three conditions we only have two conditions. So we redo this case from scratch:

$$\tilde{\phi}(k,z) = \begin{cases} C_2 e^{-kz} & \text{for} \quad 0 < z \\ B_1 e^{kz} + B_2 e^{-kz} & \text{for} \quad -d < z < 0 \\ A_1 e^{kz} & \text{for} \quad z < -d \end{cases} .$$

The conditions are

$$B_1 + B_2 = C_2 \quad , \quad A_1 e^{-kd} = B_1 e^{-kd} + B_2 e^{kd}$$

and

$$\epsilon(B_1 e^{-kd} - B_2 e^{kd}) - A_1 e^{-kd} = 0 \quad , \quad -C_2 - \epsilon(B_1 - B_2) = -4\pi/k \,.$$

We solve and get

$$\begin{aligned} C_2 &= \frac{2\pi}{k} \cdot \frac{2[\epsilon + 1 + (\epsilon - 1)e^{-2kd}]}{(\epsilon+1)^2 - (\epsilon-1)^2 e^{-2kd}} \\ A_1 &= \frac{2\pi}{k} \cdot \frac{4\epsilon}{(\epsilon+1)^2 - (\epsilon-1)^2 e^{-2kd}} \end{aligned}$$

for the field on the $z = 0$ and $z = -d$ planes. In both cases, in the far field where $k \to 0$, both just turn into $2\pi/k$ which will then simply give the bare Coulomb field $1/\rho$: as before, the slab can't screen the point charge in the far field due to the induced dipoles being localized around the charge and not having a monopolar moment. The near field $\rho \to 0$ and $k \to \infty$ for $z = 0$ surface is $C_2 = \frac{2\pi}{k} \cdot 2/(\epsilon + 1)$ which is just a Coulombic field screened by the average dielectric constant of vacuum and the slab: we have partial screening. The near field filed on the bottom slab is harder to figure out since it is a non-divergent constant so the integral just has to be done more precisely.

In the limit of a very thick slab, $d \to \infty$, $C_2 = 2\pi/k \cdot 2/(\epsilon + 1)$ so the half-screened Coulomb potential is actually the exact answer for all ρ; A_1 goes to a constant but we don't care about the potential that far away.